Interest Rate Derivatives, part 2

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Changes of measure and Girsanov Twist

- Consider the change of measure $Q(A) = E_P(N_T I(A))$ where $N_T > 0$ a.s. and $E_P N_T = 1$.

- Define $N_t = E_P(N_T | \mathcal{F}_t)$. Its a martingale. By martingale representation theorem

$$dN_t = \tilde{\nu}_t dt = \nu_t N_t dt$$

where $\nu_t = \frac{\tilde{\nu}_t}{N_t}$, which is possible because $N_t > 0$ a.s.

- Then

$$N_t = N_0 \exp(\int_0^t \nu_s dB_s - \frac{1}{2} \int_0^t \nu_2^2 ds).$$

In particular, under Q

$$dB_Q(t) = dB_t - \nu_t dt$$

is standard Brownian motion.

Equivalent Martingale Measure under arbitrary Numeraire

- We can take any positively priced security as a numeraire and denominate all other securities in terms of the chosen numeraire.
- Corresponding to each numeraire there will be a risk neutral measure
- Suppose under P, every security $\{S_t/M_t\}$ is a martingale, where M_t is the money market account
- Let $\{N_t\}$ be a strictly positive price process for a non-dividend paying asset.
- Under P, $\{N_t/M_t\}$ is a martingale.

- Consider change of measure

$$P_N(A) = E_P\left(\frac{N_T M_0}{M_T N_0} I(A)\right)$$

check that it is a valid change of measure.

- Under it each $\{S_t/N_t\}$ is a martingale. To see this, recall that if $Q(A) = E_P(Z_T I(A))$ where $Z_T > 0$ and $E_P(Z_T) = 1$,

$$E_Q(X|\mathcal{F}_t) = \frac{E_P(Z_T X|\mathcal{F}_t)}{Z_t}.$$

- Thus,

$$E_N\left(\frac{S_T}{N_T}|\mathcal{F}_t\right) = \frac{E_P \frac{N_T M_0}{M_T N_0} \frac{S_T}{N_T}}{\frac{N_t M_0}{M_t N_0}} = \frac{S_t}{N_t}.$$

The complete martingale property is similarly affirmed.

Forward rates

- Forward rate is an interest rate set today for borrowing at some date in future.
- An investor entering agreement at time t to borrow 1 at time T_1 and repay loan at time T_2, pays a continuously compounded amount of

$$\exp(f(t, T_1, T_2)(T_2 - T_1))$$

  where $f(t, T_1, T_2)$ denotes the associated forward rate.
- An arbitrage argument relates the forward rates to bond prices
- Purchase 1 bond that matures at T_1, using money from sale of x bonds maturing at time T_2. Thus,

$$B(t, T_1) = xB(t, T_2)$$

Forward rates

- I get Rs 1 at time T_1 and pay Rs. x at time T_2. Hence,

$$\exp(f(t, T_1, T_2)(T_2 - T_1)) = x = \frac{B(t, T_1)}{B(t, T_2)}.$$

- Therefore

$$f(t, T_1, T_2) = \frac{\log B(t, T_1) - \log B(t, T_2)}{T_2 - T_1}.$$

Define the instantaneous forward rate $f(t, T_1)$ as the rate as T_2 approaches T_1, then

$$f(t, T) = -\frac{\partial}{\partial T} \log B(t, T)$$

and

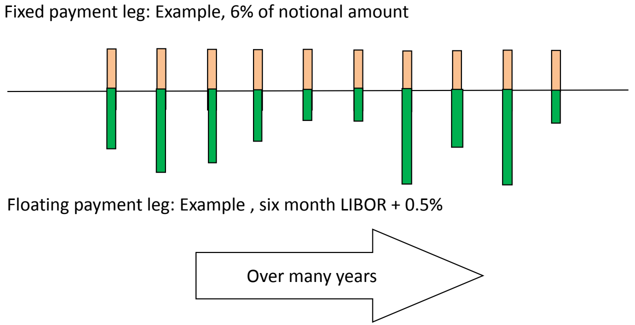
$$B(t, T) = \exp\left(-\int_t^T f(t, s) ds\right).$$

Interest rate derivatives

- We focus on pricing
  - Call and put options on discounted bonds
  - Caps and floors
  - Swaps, European and swaptions

Popular Derivatives: Swaps and Swaptions

- **Interest rate swaps** and swaptions, options on these swaps, are by far the most popular derivatives in the financial markets. The market size of these instruments was about \$310 trillion in 2007.



Gaussian short rate models

- Unlike for equities, the time horizon are much longer for interest rate derivatives
- Interest rates cannot be assumed to be constant or deterministic. They are best modelled as stochastic.
- Gaussian short rate models were the first proposed stochastic interest rate models. Stochastic yet tractable.
- We focus on the Hull and White model (1990) that continues be popular amongst practitioners

Gaussian Short Rate Models

- Vasicek Model

$$dr_t = \alpha(\theta - r_t)dt + \sigma d\tilde{B}_t$$

  Mean reverting. Highly tractable.

- Given only two parameters, poor fit to term structure of bond prices, liquid options, caps, floors.

- Ho-Lee model

$$dr_t = g(t)dt + \sigma d\tilde{B}_t$$

  $g(t)$ can be chosen to match the term structure of bond prices.

- Hull and white model discussed next is equally tractable but gives better flexibility in matching prices of liquid options

Hull and White Model

- Here, short rate, under risk neutral measure

$$dr_t = (\theta(t) - br_t)dt + \sigma d\tilde{B}_t$$

  This, generalizes the Vascizek model where $\theta(t)$ is a constant.

- Allowing $\theta(t)$ to be a function of t, allows exact calibration to zero coupon bonds

Simplifying the Bond Price

- Time t price of bond expiring at T

$$B(t,T) = \tilde{E}(\exp(-\int_t^T r_s ds)) = g(t, r_t)$$

- Let $D_t = \exp(-\int_0^t r_s ds)$ denote the discount factor. Then,

$$D_t g(t, r_t) = \tilde{E}(D_T g(T, r_T)|\mathcal{F}_t)$$

is a **martingale.**

- Expressing $D_t f(t, r_t)$ using Ito's formula, and setting the drift term to zero we get the PDE

$$g_t(t,r) + (\theta(t) - b\,r)g_r(t,r) + \frac{1}{2}\sigma^2 g_{rr}(t,r) = rg(t,r)$$

- Suppose the bond price

$$g(t,r) = e^{-A(t,T)-C(t,T)r}$$

with $A(T,T) = C(T,T) = 0$.

- Then,

$$\left((-C'(t,T) + bC(t,T) - 1)r - A'(t,T) - \theta(t)C(t,T) + \frac{1}{2}\sigma^2 C^2(t,T) \right)$$

times $g(t,r)$ equals zero.

- The two ODE's can be solved to find $A(t,T)$ and $C(t,T)$.

- From

$$-C'(t,T) + bC(t,T) - 1 = 0$$

We get

$$C(t,T) = \frac{1 - e^{-b(T-t)}}{b}.$$

- We also have

$$A'(t,T) = -\theta(t)C(t,T) + \frac{1}{2}\sigma^2 C^2(t,T).$$

Integrating, with respect to t,

$$A(t,T) = \int_t^T \left(\theta(s)C(s,T) - \frac{1}{2}\sigma^2 C^2(s,T)\right) ds$$

Simplifying $A(t, T)$ to help in calibration

- Double differentiating w.r.t. T, with some manipulations, we get

$$\theta(T) = bf(0, T) - \frac{\partial}{\partial T} f(0, T) + \frac{\sigma^2}{2b}(1 - e^{bT})$$

where, the well known forward rate

$$f(t, T) = -\frac{\partial}{\partial T} \log B(t, T).$$

Recall that $B(t, T)$ denotes the price of a bond at time t that gives Rs.1 at time T.

- Plugging $\theta(t)$ in $A(t, T)$ expression, we get

$$A(t, T) = \log \frac{B(0, T)}{B(0, t)} + C(t, T) f(0, t) - \frac{\sigma^2}{2b}(1 - e^{-b(T-t)}) C(t, T)^2.$$

Distribution of the short rate

- Recall, under risk neutral measure

$$dr_t = (\theta(t) - br_t)dt + \sigma dB_t$$

- Solving for r_t, consider

$$d\left(e^{bu} r_u\right) = e^{bu}(br_u du + dr_u) = e^{bu}(\theta(u)du + \sigma dB_u)$$

- Equivalently,

$$e^{bT} r_T = e^{bt} r_t + \int_t^T e^{bu} \theta(u) du + \sigma \int_t^T e^{bu} dB_u$$

- Or

$$r_T = e^{-b(T-t)} r_t + \int_t^T e^{-b(T-u)} \theta(u) du + \sigma \int_t^T e^{-b(T-u)} dB_u$$

- Plugging in earlier expression for $\theta(u)$ above, and simplifying we get

$$r_T = e^{-b(T-t)} r_t + \alpha(T) - e^{-b(T-t)} \alpha(t) + \sigma \int_t^T e^{-b(T-u)} dB_u$$

where

$$\alpha(t) = f(0,t) + \frac{\sigma^2}{2b^2}(1 - e^{-bt})^2$$

Define

$$dx_t = -bx_t dt + \sigma dB_t, \quad x_0 = 0.$$

Then,

$$x_T = e^{-b(T-t)} x_t + \sigma \int_t^T e^{-b(T-u)} dB_u$$

so that

$$r_t = x_t + \alpha_t$$

for all t. It has a Gaussian distribution!

Generating samples for r_t

- $$dx_t = -bx_t dt + \sigma dB_t, \quad x_0 = 0.$$
  is easy to simulate using Monte Carlo.

- Observe that
  $$x_t = x_s e^{-b(t-s)} + \sigma \int_s^t e^{-b(t-u)} dB(u)$$

- Thus, x_t is Gaussian with mean
  $$E(x_t|\mathcal{F}_s) = x_s e^{-b(t-s)}$$
  and
  $$Var(x_t|\mathcal{F}_s) = \frac{\sigma^2}{2b}.$$

Generating samples for r_t

- Further,

$$\alpha(t) = f(0,t) + \frac{\sigma^2}{2b^2}(1 - e^{-bt})^2$$

is easier to calibrate accurately compared to

$$\theta(T) = bf(0,T) - \frac{\partial}{\partial T} f(0,T) + \frac{\sigma^2}{2b}(1 - e^{-b(T-t)}).$$

- Thus, no need to find $\{\theta_t\}$ for calibration or for simulation
- The sampling algorithm involves generating samples of $x(t)$ at discrete points and adding $\alpha(t)$ to them at those points to get samples of r_t.